

# Borel summability of perturbative series in 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ theories

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(Dated: May 10, 2016)

We study weak coupling perturbative series in 4d  $\mathcal{N} = 2$  and 5d  $\mathcal{N} = 1$  supersymmetric gauge theories with Lagrangians. We prove that the perturbative series of these theories in zero instanton sector are Borel summable for various observables. Our result for 4d  $\mathcal{N} = 2$  case supports an expectation from a recent proposal on a semiclassical realization of infrared renormalons in QCD-like theories, where the semiclassical solution does not exist in  $\mathcal{N} = 2$  theories and the perturbative series are unambiguous, namely Borel summable. We also prove that the perturbative series in arbitrary number of instanton sector are Borel summable for a wide class of theories. It turns out that exact results can be obtained by summing over the Borel resummations in each number of instanton sector.

## I. INTRODUCTION

Weak coupling perturbation theory in quantum field theory (QFT) typically yields asymptotic series [1]. One of standard ways to resum asymptotic series is Borel resummation: when we have a perturbative series  $I(g) = \sum_{\ell=0}^{\infty} c_{\ell} g^{a+\ell}$ , Borel summation is given by

$$\int_0^{\infty} dt e^{-\frac{t}{g}} \mathcal{B}I(t), \quad (1)$$

where  $\mathcal{B}I(t)$  is analytic continuation of the Borel transformation  $\sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\Gamma(a+\ell)} t^{a+\ell-1}$  after taking the summation. However, it is expected that perturbative series in typical interacting QFT is not Borel summable due to singularities at positive real axis in  $\mathcal{B}I(t)$  [2].

Nevertheless one might wonder if situation becomes better in theories with huge symmetries such as extended supersymmetric (SUSY) theories. Indeed some SUSY theories have some protected quantities, which do not receive quantum corrections or receive only one-loop correction. On the other hand it is totally unclear for unprotected observables, which receive higher loop perturbative corrections and non-perturbative corrections, when perturbative series are convergent, Borel summable or non-Borel summable.

Recently there appeared an interesting hint [3] to this question from resurgence approach to asymptotic free gauge theory on  $S^1 \times \mathbb{R}^3$  (see also [4]). In [3] the authors have proposed a semiclassical realization of infrared (IR) renormalons, which correspond to leading singularities in Borel plane of QCD-like theories. In this proposal, the IR renormalons are realized by so-called bion-anti-bion contributions [5] and cancel ambiguities of resummation in perturbation theory. This mechanism is QFT analogue of Bogomolny-Zinn-Justin mechanism in quantum mechanics [6]. While this scenario is expected for non-SUSY theory and  $\mathcal{N} = 1$  theory, there are no bion solutions in theories with  $\mathcal{N} \geq 2$  SUSY [7]. Therefore it

is expected that perturbative expansions themselves in  $\mathcal{N} \geq 2$  theories do not have ambiguities and are Borel summable. However there are very few examples to support this expectation so far, which are  $S^4$  partition functions and SUSY Wilson loops in  $SU(2), U(2)$  gauge theories [8, 9], and extremal correlators in superconformal QCD with  $SU(2)$  and  $SU(3)$  gauge groups [10] (see also [11]).

Main purpose of this paper is to provide strong evidence for this expectation. Namely we prove Borel summability [30] of perturbative series in general 4d  $\mathcal{N} = 2$  theories with Lagrangians for various observables [31]. Our main tool is the localization method [13], which reduces path integrals to finite dimensional integrals for a class of observables. We also prove Borel summability in arbitrary number of instanton sector when we know explicit expressions of instanton corrections to localization formula, which are captured by so-called Nekrasov instanton partition function [14]. In section II we give proofs for  $S^4$  partition function, SUSY Wilson loop, Bremsstrahlung function, extremal correlator and squashed  $S^4$  partition function. As a conclusion our result strongly supports the expectation from the proposal [3] on the semiclassical realization of IR renormalons in QCD-like theories.

Another main result of this paper is that the Borel resummation in a fixed number of instanton sector is exactly the same as the truncation of the full result to this sector. This means that perturbative series in each number of instanton sector does not mix with each other from the viewpoint of resummation. This feature is quite different from successful examples of resurgence approach, where perturbative expansion in zero instanton sector is related to ones in non-zero instanton sector. We discuss this point in more detail in sec. IV.

In section III we also discuss 5d  $\mathcal{N} = 1$  gauge theories. While 5d gauge theory is not renormalizable in the sense of power counting, one can reasonably compute some observables in UV complete theories [15]. We can prove Borel summability of perturbative series for squashed  $S^5$  partition function and SUSY Wilson loop.

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## II. 4D $\mathcal{N} = 2$ THEORY

### A. $S^4$ partition function

We begin with partition function on  $S^4$ . Let us consider 4d  $\mathcal{N} = 2$  theory with semi-simple gauge group  $G = G_1 \times \cdots \times G_n$  and  $N_f$  hyper multiplets of representations  $(\mathbf{R}_1, \cdots, \mathbf{R}_{N_f})$ . Thanks to the localization method, the  $S^4$  partition function can be expressed in terms of the following finite dimensional integral [13]

$$Z_{S^4} = \int_{-\infty}^{\infty} d^{|G|} a \ Z_{\text{VdM}} Z_{\text{cl}} Z_{1\text{loop}} Z_{\text{inst}}, \quad (2)$$

where  $a$  takes values in Cartan subalgebra of  $G$ .  $Z_{\text{cl}}$  and  $Z_{1\text{loop}}$  are classical and one-loop contributions around localization locus, respectively [32]:

$$\begin{aligned} Z_{\text{VdM}}(a) &= \prod_{\alpha \in \text{root}_+} (\alpha \cdot a)^2, \\ Z_{\text{cl}}(a) &= \exp \left[ - \sum_{p=1}^n \frac{1}{g_p} \text{tr}(a^{(p)})^2 \right], \\ Z_{1\text{loop}}(a) &= \frac{\prod_{\alpha \in \text{root}_+} H^2(\alpha \cdot a)}{\prod_{m=1}^{N_f} \prod_{\rho_m \in \mathbf{R}_m} H(\rho_m \cdot a)}, \\ H(x) &= e^{-(1+\gamma)x^2} G(1+ix) G(1-ix), \end{aligned} \quad (3)$$

where the parameter  $g_p$  is proportional to square of Yang-Mills coupling of the gauge group  $G_p$ ,  $\rho_m$  is weight vector of the representation  $\mathbf{R}_m$ ,  $\gamma$  is Euler constant and  $G(x)$  is Barnes  $G$ -function.  $Z_{\text{inst}}$  is contributions from instantons, which are described by Nekrasov instanton partition function  $Z_{\text{Nek}}(a)$  [14] with  $\Omega$ -background parameters  $\epsilon_1 = \epsilon_2 = 1$  [33] as

$$Z_{\text{inst}}(a) = |Z_{\text{Nek}}(a)|^2 = \sum_{\{k_p\}=0}^{\infty} e^{-\sum_{p=1}^n \frac{k_p}{g_p}} Z_{\text{inst}}^{(k_1, \dots, k_n)}(a). \quad (4)$$

Now we are interested in weak coupling expansion of  $Z_{S^4}$  in a fixed number of instanton sector:

$$Z_{S^4}^{(k_1, \dots, k_n)}(g) = \int_{-\infty}^{\infty} d^{|G|} a \ Z_{\text{VdM}} Z_{\text{cl}} Z_{1\text{-loop}} Z_{\text{inst}}^{(k_1, \dots, k_n)}. \quad (5)$$

This has the following weak coupling expansion:

$$Z_{S^4}^{(k_1, \dots, k_n)} \sim \sum_{\{\ell_p\}=0}^{\infty} c_{\ell_1, \dots, \ell_n}^{(k_1, \dots, k_n)} \prod_{p=1}^n g_p^{\frac{\dim(G_p)}{2} + \ell_p}. \quad (6)$$

Here we prove that the small- $g$  expansion of  $Z_{S^4}^{(k_1, \dots, k_n)}(g)$  is Borel summable and this is true also for various observables. It is also interesting to ask property of the expansion by instanton number as in [9] but this is beyond scope of this paper.

### $SU(N)$ superconformal QCD

First we focus on the 4d  $\mathcal{N} = 2$   $SU(N)$  superconformal QCD with  $2N$  fundamental hyper multiplets. We will consider more general theory later. The  $S^4$  partition function of this theory in  $k$ -instanton sector is given by

$$\begin{aligned} Z_{\text{SQCD}}^{(k)} &= \int_{-\infty}^{\infty} d^N a \ \delta\left(\sum_j a_j\right) \prod_{i < j} (a_i - a_j)^2 \\ &\quad e^{-\frac{1}{g} \sum_j a_j^2} \frac{\prod_{i < j} H^2(a_i - a_j)}{\prod_j H^{2N}(a_j)} Z_{\text{inst}}^{(k)}(a), \end{aligned} \quad (7)$$

where the delta function comes from speciality of  $SU(N)$ . We would like to know property of small- $g$  expansion of  $Z_{\text{SQCD}}^{(k)}$ . For this purpose let us take the coordinate

$$a_i = \sqrt{t} \hat{x}_i, \quad (8)$$

where  $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_N)$  is the unit vector spanning unit  $S^{N-1}$ . Then we rewrite the partition function as [34]

$$Z_{\text{SQCD}}^{(k)} = \int_0^{\infty} dt \ e^{-\frac{t}{g}} f^{(k)}(t), \quad (9)$$

where

$$\begin{aligned} f^{(k)}(t) &= \frac{t^{\frac{N^2-3}{2}}}{2} \int_{S^{N-1}} d^{N-1} \hat{x} \ \delta\left(\sum_j \hat{x}_j\right) h^{(k)}(t, \hat{x}), \\ h^{(k)}(t, \hat{x}) &= Z_{\text{VdM}}(\hat{x}) Z_{1\text{-loop}}(\sqrt{t} \hat{\mathbf{x}}) Z_{\text{inst}}^{(k)}(\sqrt{t} \hat{\mathbf{x}}). \end{aligned} \quad (10)$$

Note that this takes the form of the Laplace transformation as in the Borel resummation formula (1). So it would be natural to expect that  $f^{(k)}(t)$  is Borel transformation of the original perturbative series, namely analytic continuation of  $\sum_{\ell=0}^{\infty} \frac{c_{\ell}^{(k)}}{\Gamma(\frac{N^2-1+\ell}{2})} t^{\ell}$  [35]. We will prove this in the following steps: (I) We show that the integrand  $h^{(k)}(t, \hat{x})$  is identical to analytic continuation of a convergent power series of  $t$ . (II) We ask if we can exchange the power series expansion of  $h^{(k)}(t, \hat{x})$  and integration over  $\hat{x}$ . We show this by proving uniform convergence of the small- $t$  expansion. (III) The Laplace transformation (9) guarantees that the coefficient of the perturbative series of  $f^{(k)}(t)$  at  $\mathcal{O}(t^{\frac{N^2-3}{2}+\ell})$  is given by  $c_{\ell}^{(k)}/\Gamma(\frac{N^2-1+\ell}{2})$ .

For simplicity we first focus on zero-instanton sector and non-zero instanton sector will be considered later. By using product representation of the Barnes  $G$ -function [36],  $h^{(0)}(t, \hat{x})$  can be written as

$$h^{(0)}(t, \hat{x}) = Z_{\text{VdM}}(\hat{x}) \prod_{n=1}^{\infty} \frac{\prod_{i < j} \left(1 + \frac{t(\hat{x}_i - \hat{x}_j)^2}{n^2}\right)^{2n}}{\prod_j \left(1 + \frac{t\hat{x}_j^2}{n^2}\right)^{2Nn}}. \quad (11)$$

Plugging  $\sum_{n=1}^{\infty} n \log \left(1 + \frac{x^2}{n^2}\right) = - \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \zeta(2\ell-1)}{\ell} x^{2\ell}$  into this, we find generating function for the small- $t$  ex-

pansion of  $h^{(0)}$  as

$$Z_{VdM}(\hat{x}) \exp \left[ -2 \sum_{i < j} \sum_{\ell=2}^{\infty} \frac{(-t)^\ell \zeta(2\ell-1)}{\ell} (\hat{x}_i - \hat{x}_j)^{2\ell} + 2N \sum_j \sum_{\ell=2}^{\infty} \frac{(-t)^\ell \zeta(2\ell-1)}{\ell} \hat{x}_j^{2\ell} \right]. \quad (12)$$

The small- $t$  expansion of  $h^{(0)}$  has a finite radius of convergence  $t_0$ , which is dependent on  $\hat{x}$  but larger than 1. Hence  $h^{(0)}$  is the same as analytic continuation of the convergent power series of  $t$ .

Next we show commutativity of the summation and integration over  $\hat{x}$  by proving uniform convergence of the small- $t$  expansion of  $h^{(0)}$ . For this purpose it is convenient to apply Weierstrass's M-test. Namely we find a sequence  $\{M_\ell\}$  satisfying  $|h_\ell^{(0)}(\hat{x})| < M_\ell$  and  $\sum_{\ell=0}^{\infty} M_\ell < \infty$  for fixed  $t$ . We can easily find such a series. For example, since  $\zeta(2\ell-1) < 2$ ,  $|\hat{x}| \leq 1$ , a generating function  $\bar{h}^{(0)}(t)$  of  $M_\ell$  can be obtained by the replacements  $(-1)^{\ell+1} \zeta(2\ell-1) (\hat{x}_i - \hat{x}_j)^{2\ell} \rightarrow 2$ ,  $(-1)^\ell \zeta(2\ell-1) \hat{x}_j^{2\ell} \rightarrow 2$  in (12):

$$\bar{h}^{(0)}(t) = \exp \left[ 2N(3N-1) \sum_{\ell=2}^{\infty} \frac{t^\ell}{\ell} \right] = \left( \frac{e^{-t}}{1-t} \right)^{2N(3N-1)}. \quad (13)$$

Thus  $f^{(0)}(t)$  is actually the Borel transformation of the original perturbative series.

To show Borel summability of the perturbative series we should ask analytic property of the Borel transformation  $f^{(0)}(t)$ . First  $f^{(0)}(t)$  does not have branch cut along  $t \in \mathbb{R}_+$  though this has a branch cut along  $t \in \mathbb{R}_-$  for even  $N$  [37]. Structure of poles can be easily seen from the infinite product representation (11) of  $h^{(0)}$ . Since  $h^{(0)}$  cannot have poles unless  $t \in \mathbb{R}_-$  and integration is finite dimensional over compact region,  $f^{(0)}$  does not have poles along positive real axis. Thus, we conclude that the perturbative series of  $Z_{\text{SQCD}}$  in zero instanton sector is Borel summable. As we discuss below, generalization to other theories and observables, and nonzero instanton sector are quite straightforward.

#### General theory with Lagrangian

Generalization to other theories is quite parallel to the case of the SQCD. First we insert delta function constraint  $\Delta(a)$  to the integrand as in (7) such that the following coordinate

$$a_i^{(p)} = \sqrt{t_p} \hat{x}_i^{(p)}, \quad (14)$$

parametrizes sphere with the radius  $\sqrt{t_p}$ . Then the partition function again takes the form of Laplace transformation with multi variables:

$$Z^{(k_1, \dots, k_n)} = \int_0^\infty d^n t \, e^{-\sum_{p=1}^n \frac{t_p}{g_p}} f^{(k_1, \dots, k_n)}(t), \quad (15)$$

where

$$f^{(k_1, \dots, k_n)}(t) = \frac{t^{\frac{\dim(G)}{2}-1}}{2^n} \int_{\text{sphere}} d\hat{x} \, \Delta(\hat{x}) h^{(k_1, \dots, k_n)}(t, \hat{x}),$$

$$h^{(k_1, \dots, k_n)}(t) = Z_{\text{VdM}}(\hat{x}) Z_{1\text{loop}} Z_{\text{inst}}|_{a_i^{(p)} = \sqrt{t_p} \hat{x}_i^{(p)}}, \quad (16)$$

with  $t^{\frac{\dim(G)}{2}-1} = \prod_{p=1}^n t_p^{\frac{\dim(G_p)}{2}-1}$ . Let us focus on zero-instanton sector again. Then we can always prove that  $h^{(k_1, \dots, k_n)}$  with  $k_p = 0$  gives a uniform convergent power series of  $t$ . Namely we can always construct convergent series as in (13) to pass Weierstrass's M-test. Hence  $f^{(k_1, \dots, k_n)}(t)$  at zero instanton sector is actually Borel transformation. The Borel transformation cannot have poles and branch cut along positive real axis. Therefore the perturbative series of  $Z_{S^4}$  in zero instanton sector is Borel summable for general  $\mathcal{N} = 2$  theory with Lagrangian.

#### Nonzero instanton sector

Generalization to arbitrary number of instanton sector is also straightforward when we know explicit forms of Nekrasov partition functions. This is because  $Z_{\text{inst}}^{(k_1, \dots, k_n)}(a)$  for all the known cases is rational function of  $a$ , whose poles are not located at real axis. For example, Nekrasov partition function of  $U(N)$  SQCD with  $N_f$  fundamentals and anti-fundamentals is given by [38]

$$Z_{\text{Nek, SQCD}}(a) = \sum_{\{Y_1, \dots, Y_N\}} q^{|Y|} \frac{\left( \prod_{j=1}^N n_j^f(a, Y) \right)^{N_f}}{\prod_{i,j=1}^N n_{i,j}^V(a, Y)}, \quad (17)$$

where  $Y_j$  is Young diagram associated with  $a_j$  and

$$n_{i,j}^V(a, Y) = \prod_{s \in Y_i} E_{ij}(a, s) (\epsilon_1 + \epsilon_2 - E_{ij}(a, s)),$$

$$E_{ij}(a, s) = -\epsilon_1 A_{Y_j}(s) + \epsilon_2 (L_{Y_i}(s) + 1) - i(a_i - a_j),$$

$$n_j^f(a, Y) = \prod_{s \in Y_j} \phi_j(a, s) (\phi_j(a, s) + \epsilon_1 + \epsilon_2),$$

$$\phi_j(a, s) = -ia_j + \epsilon_1(s_h - 1) + \epsilon_2(s_v - 1). \quad (18)$$

Here  $s = (s_h, s_v)$  labels box in Young tableau at  $s_h$ -th column and  $s_v$ -th row, and  $L_Y$  ( $A_Y$ ) is leg (arm) length of Young tableau  $Y$  at  $s$ . Although the contribution from each Young diagram may have poles along real axis of  $a$ , poles of this type are canceled after summing over Young diagrams with the same instanton number [13]. This is true unless  $m_1\epsilon_1 + m_2\epsilon_2$  ( $m_{1,2} \in \mathbb{Z}$ ) can be purely imaginary but  $Z_{S^4}$  for this case is ill-defined and therefore we do not consider this case. This feature is common among all the cases, where expressions of Nekrasov partition function are explicitly known.

Keeping this in mind, now we can easily prove Borel summability as in zero instanton sector. Since  $h^{(k_1, \dots, k_n)}$  is product of  $h^{(0, \dots, 0)}$  and rational function of  $\sqrt{t_p}$ , small- $t$

expansion of  $h^{(k_1, \dots, k_n)}$  is always uniform convergent and hence  $f^{(k_1, \dots, k_n)}(t)$  is always Borel transformation of the original perturbative series. We can easily see that the Borel transformation cannot have poles and branch cuts along positive real axis by using the above property of  $Z_{\text{inst}}$ . Thus the perturbative series of  $Z_{S^4}$  in arbitrary number of instanton sector is Borel summable.

## B. Other observables

### Supersymmetric Wilson loop

Generalization to some other observables is straightforward as well. First let us consider the Wilson loop

$$W_{\mathbf{R}}(C) = \text{tr}_{\mathbf{R}} P \exp \left[ i \oint_C ds (A_\mu \dot{x}^\mu + i\Phi) \right], \quad (19)$$

where  $\Phi$  is the adjoint scalar in vector multiplet. The Wilson loop is supersymmetric when the contour  $C$  is the grand circle of  $S^4$ . By applying the localization method, VEV of the Wilson loop is represented by the following VEV of the matrix model

$$\langle W_{\mathbf{R}}(\text{Circle}) \rangle = \langle \text{tr}_{\mathbf{R}} e^a \rangle_{\text{M.M.}} \quad (20)$$

Since this is just finite linear combination of exponentials, this does not give anything harmful. Thus repeating the above arguments, we can prove that perturbative series of the SUSY Wilson loop is Borel summable. Obviously products of Wilson loops also give Borel summable perturbative series.

### Bremsstrahlung function in SCFT

Bremsstrahlung function  $B$  appears in cusp anomalous dimension of small boost:  $\Gamma_{\text{cusp}}(\varphi) = B\varphi^2 + \mathcal{O}(\varphi^4)$ , where  $\varphi$  is the boost parameter. This determines an energy radiated by accelerating quarks [17]. It was conjectured that the Bremsstrahlung function in  $\mathcal{N} = 2$  superconformal theory is given by the following VEV of the matrix model [18]

$$B = \frac{1}{4\pi^2} \frac{\partial}{\partial b} \langle \text{tr} e^{ba} \rangle_{\text{M.M.}} \Big|_{b=1}, \quad (21)$$

which is formally derivative of the supersymmetric Wilson loop in fundamental representation with winding number  $b$ . Since we have shown Borel summability for the Wilson loop,  $B$  is also Borel summable.

### Extremal correlator in SCFT

Next we consider the correlation function

$$\langle \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n) \overline{\mathcal{O}}_{\bar{J}}(y) \rangle, \quad (22)$$

where  $\mathcal{O}_I$  and  $\overline{\mathcal{O}}_{\bar{I}}$  are chiral and anti-chiral primary operators, respectively. This is often called extremal correlator. It is known that this is determined by the two point function  $\langle \mathcal{O}_I(x) \overline{\mathcal{O}}_{\bar{J}}(y) \rangle$ . It was shown [10] that the two point function is given by a ratio of (finite) linear combination of the quantity  $\langle \prod_j (\text{tr} a^{m_j})^{n_j} \rangle_{\text{M.M.}}$ . We can show Borel summability for this quantity by repeating the above arguments and hence perturbative series for the extremal correlator is also Borel summable.

### Squashed $S^4$ partition function

Next let us consider partition function  $Z_{S_b^4}$  on squashed  $S^4$  with a squashing parameter  $b$ . This has a simple relation to supersymmetric Renyi entropy [19, 20]. There are two differences from the round  $S^4$  partition function. One is one-loop determinant [21, 22]:

$$Z_{1\text{loop}}(a) = \frac{\prod_{\alpha \in \Delta_+} \Upsilon(ia \cdot \alpha) \Upsilon(-ia \cdot \alpha) / (\alpha \cdot a)^2}{\prod_{m=1}^{N_f} \prod_{\rho_m \in \mathbf{R}_m} \Upsilon\left(ia \cdot \rho + \frac{Q}{2}\right)}, \quad (23)$$

where  $Q = b + b^{-1}$  and

$$\Upsilon(x) = \prod_{m_1, m_2 \geq 0} (m_1 b + m_2 b^{-1} + x)(m_1 b + m_2 b^{-1} + Q - x). \quad (24)$$

The other is values of  $\Omega$ -background parameters in  $Z_{\text{Nek}}$ , which are  $\epsilon_1 = b, \epsilon_2 = b^{-1}$ .

Repeating the argument for  $Z_{S^4}$ , we can always rewrite  $Z_{S_b^4}$  as the Laplace transformation as in (15) and show that  $f^{(k_1, \dots, k_n)}(t)$  for  $Z_{S_b^4}$  is Borel transformation of original perturbative series. However, pole structure of the Borel transformation is slightly more involved. This depends on  $b$  and we have poles in real positive axis when  $m_1 b + m_2 b^{-1}$  ( $m_{1,2} \in \mathbb{Z}$ ) can be purely imaginary [39]. Because the partition function for this region is ill-defined, we conclude that  $Z_{S_b^4}$  gives Borel summable perturbative series when it is well-defined.

## III. 5d $\mathcal{N} = 1$ THEORY

We also study perturbative series of 5d  $\mathcal{N} = 1$  SUSY theory. First we study squashed  $S^5$  partition function with squashing parameters  $(\phi_1, \phi_2, \phi_3)$ , which has a simple relation to SUSY Renyi entropy [23]. We can show Borel summability of perturbative series in a quite parallel way to the 4d case. The  $S^5$  partition function can be computed by localization [24] and this takes the form of (2). While classical part is the same up to redefinition of  $g$  [40], one-loop part is given by

$$Z_{1\text{loop}}(a) = \frac{\prod_{\alpha \in \text{root}} S_3(-i\alpha \cdot a; \vec{\omega}) / (\alpha \cdot a)}{\prod_{m=1}^{N_f} \prod_{\rho_m \in \mathbf{R}_m} S_3(-i\rho_m \cdot a + \frac{\omega_1 + \omega_2 + \omega_3}{2}; \vec{\omega})}, \quad (25)$$



where  $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$  with  $\omega_j = 1 + \phi_j$  and

$$S_3(z; \vec{\omega}) = \prod_{n_1, n_2, n_3 \geq 0} (\vec{n} \cdot \vec{\omega} + z) \prod_{n_1, n_2, n_3 \geq 1} (\vec{n} \cdot \vec{\omega} - z), \quad (26)$$

with  $\vec{n} \cdot \vec{\omega} = n_1\omega_1 + n_2\omega_2 + n_3\omega_3$ . Instanton contribution for this case is product of three 5d Nekrasov partition functions with  $\Omega$ -background parameters  $(\epsilon_1, \epsilon_2) = (\phi_2 - \phi_1, \phi_3 - \phi_1)$ ,  $(\phi_3 - \phi_2, \phi_1 - \phi_2)$  and  $(\phi_1 - \phi_3, \phi_2 - \phi_3)$ . Nekrasov partition function in 5d is not rational function but ratio of hyperbolic functions. As in 4d case, the 5d Nekrasov partition function does not have poles in real axis unless  $m_1\epsilon_1 + m_2\epsilon_2$  ( $m_{1,2} \in \mathbb{Z}$ ) can be purely imaginary. Since the  $S^5$  partition function is ill-defined when  $\vec{n} \cdot \vec{\omega} + (\omega_1 + \omega_2 + \omega_3)/2$  and  $m_1\epsilon_1 + m_2\epsilon_2$  can be purely imaginary, we do not consider these cases.

Now we can prove Borel summability for the  $S^5$  partition function as in 4d. First by the transformation (14), we can rewrite the partition function in the form of Laplace transformation as in (15). A similar argument shows that the integrand is again Borel transformation. Then as long as the partition function is well-defined, the Borel transformation does not have singularity along positive real axis of Borel plane. Thus we conclude that the perturbative series of the  $S^5$  partition function is Borel summable for arbitrary number of instanton sector.

We can also show Borel summability for supersymmetric Wilson loop on round  $S^5$ . The Wilson loop of the type (19) is supersymmetric when its contour is Hopf fibre at one point of  $\mathbb{CP}^2$  base. The result of localization is

$$\langle W_{\mathbf{R}}(\text{Hopf fibre}) \rangle = \langle \text{tr} \mathbf{R} e^{2\pi a} \rangle_{\text{M.M.}} \quad (27)$$

Insertion of this operator does not spoil our argument as in 4d and thus perturbative series of the SUSY Wilson loop is also Borel summable.

#### IV. DISCUSSIONS

We have studied the weak coupling perturbative series in 4d  $\mathcal{N} = 2$  and 5d  $\mathcal{N} = 1$  SUSY gauge theories with Lagrangians. We have proven Borel summability of the perturbative expansions in zero instanton sector for various observables. We have also proven Borel summability in arbitrary number of instanton sector when we know explicit forms of Nekrasov partition functions. Thus our result is nontrivially consistent with the proposal [3] on the semiclassical realization of IR renormalons.

Our result also shows that the Borel resummation in a fixed number of instanton sector is exactly the same as the truncation of the full result to this sector. There are two conceptually important implications of this. First, we can obtain the exact results by summing over the Borel resummation in each number of instanton sector. If this is true for all physical observables in 4d  $\mathcal{N} = 2$

theories, then one can define 4d  $\mathcal{N} = 2$  theories in this way although we have not proven it. We leave this for our future problem.

Second, our result means that the Borel resummation in the zero instanton sector *does not give* the full result including instanton corrections and perturbative series in each number of instanton sector is “isolated” in some sense. Thus perturbative data in different numbers of instanton sector do not mix with each other. This feature was already pointed out in [9] for  $SU(2)$  and  $U(2)$  cases. Our result says that this is common for quite general 4d  $\mathcal{N} = 2$  and 5d  $\mathcal{N} = 1$  theories. Note also that this feature itself was observed long time ago in Seiberg-Witten prepotential [25], which receives only one-loop perturbative corrections but has instanton corrections. A significant difference from our examples is that the perturbative correction in the prepotential is trivially Borel summable while our examples generally have factorially divergent perturbative expansions, whose Borel summabilities were a priori nontrivial. Similar behaviors appear also in WKB quantization of the quartic oscillator and  $1/N$ -expansion of partition function of ABJM theory on  $S^3$  [26], for example (see also [27]). The above feature is different from successful examples of resurgence approach, where perturbative data in zero instanton sector is related to ones in non-zero instanton sector. We shall ask if this feature is common for less SUSY theories or not.

It is interesting to consider perturbative series of ’t Hooft loop in 4d  $\mathcal{N} = 2$  theory, whose localization formula has been obtained in [28]. Apparently it seems more involved because the ’t Hooft loop receives corrections from monopoles as well as instantons.

It would be also illuminating to study perturbative series of 3d CS matter theories on  $S^3$  by inverse of CS level as in [8, 9]. We cannot naively apply our argument to these theories because exponential factor is purely imaginary for these theories. Hence we need to change integral contour to get usual Laplace transformation but this change picks up residues from poles of integrand. Probably we should think of it more carefully.

We close by mentioning that our result would be closely related to a connection between planar limit and “very strong coupled large- $N$  limit” discussed in [29]. It is attractive if one can make it more precise from our viewpoint.

#### Acknowledgments

We thank Zohar Komargodski for helpful discussions and comments on the draft. We are grateful to Jorge G. Russo, Ricardo Schiappa, Yuji Tachikawa and anonymous Referees of PRL for useful comments on the draft.

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- [1] F. J. Dyson, “Divergence of perturbation theory in quantum electrodynamics,” *Phys. Rev.* **85**, 631 (1952).
- [2] G. ’t Hooft, “Can We Make Sense Out of Quantum Chromodynamics?,” *Subnucl. Ser.* **15**, 943 (1979).
- [3] P. Argyres and M. Unsal, “A semiclassical realization of infrared renormalons,” *Phys. Rev. Lett.* **109**, 121601 (2012) [arXiv:1204.1661 [hep-th]], “The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects,” *JHEP* **1208**, 063 (2012) [arXiv:1206.1890 [hep-th]].
- [4] G. V. Dunne and M. Unsal, “Resurgence and Trans-series in Quantum Field Theory: The  $CP(N-1)$  Model,” *JHEP* **1211**, 170 (2012) [arXiv:1210.2423 [hep-th]], “Continuity and Resurgence: towards a continuum definition of the  $CP(N-1)$  model,” *Phys. Rev. D* **87**, 025015 (2013) [arXiv:1210.3646 [hep-th]]; A. Cherman, D. Dorigoni, G. V. Dunne and M. Unsal, “Resurgence in Quantum Field Theory: Nonperturbative Effects in the Principal Chiral Model,” *Phys. Rev. Lett.* **112**, 021601 (2014) [arXiv:1308.0127 [hep-th]]; T. Misumi, M. Nitta and N. Sakai, “Neutral bions in the  $CP^{N-1}$  model,” *JHEP* **1406**, 164 (2014) [arXiv:1404.7225 [hep-th]].
- [5] M. Unsal, “Magnetic bion condensation: A New mechanism of confinement and mass gap in four dimensions,” *Phys. Rev. D* **80**, 065001 (2009) [arXiv:0709.3269 [hep-th]].
- [6] E. B. Bogomolny, “Calculation Of Instanton - Anti-instanton Contributions In Quantum Mechanics,” *Phys. Lett. B* **91**, 431 (1980); J. Zinn-Justin, “Multi - Instanton Contributions in Quantum Mechanics,” *Nucl. Phys. B* **192**, 125 (1981).
- [7] E. Poppitz and M. Unsal, “Seiberg-Witten and ‘Polyakov-like’ magnetic bion confinements are continuously connected,” *JHEP* **1107**, 082 (2011) [arXiv:1105.3969 [hep-th]].
- [8] J. G. Russo, “A Note on perturbation series in supersymmetric gauge theories,” *JHEP* **1206**, 038 (2012) [arXiv:1203.5061 [hep-th]].
- [9] I. Aniceto, J. G. Russo and R. Schiappa, “Resurgent Analysis of Localizable Observables in Supersymmetric Gauge Theories,” *JHEP* **1503**, 172 (2015) [arXiv:1410.5834 [hep-th]].
- [10] E. Gerchkovitz, J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. S. Pufu, “Correlation Functions of Coulomb Branch Operators,” arXiv:1602.05971 [hep-th].
- [11] M. Baggio, V. Niarchos and K. Papadodimas, “Exact correlation functions in  $SU(2)_{\mathcal{N}=2}$  superconformal QCD,” *Phys. Rev. Lett.* **113**, no. 25, 251601 (2014) [arXiv:1409.4217 [hep-th]], “ $tt^*$  equations, localization and exact chiral rings in 4d  $\mathcal{N}=2$  SCFTs,” *JHEP* **1502**, 122 (2015) [arXiv:1409.4212 [hep-th]].
- [12] G. ’t Hooft, “On the Convergence of Planar Diagram Expansions,” *Commun. Math. Phys.* **86**, 449 (1982).
- [13] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” *Commun. Math. Phys.* **313**, 71 (2012) [arXiv:0712.2824 [hep-th]].
- [14] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7**, no. 5, 831 (2003) [hep-th/0206161]; N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” *Prog. Math.* **244**, 525 (2006) [hep-th/0306238].
- [15] N. Seiberg, “Five-dimensional SUSY field theories, non-trivial fixed points and string dynamics,” *Phys. Lett. B* **388**, 753 (1996) [hep-th/9608111].
- [16] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” *Lett. Math. Phys.* **91**, 167 (2010) [arXiv:0906.3219 [hep-th]].
- [17] D. Correa, J. Henn, J. Maldacena and A. Sever, “An exact formula for the radiation of a moving quark in  $N=4$  super Yang Mills,” *JHEP* **1206**, 048 (2012) [arXiv:1202.4455 [hep-th]].
- [18] B. Fiol, E. Gerchkovitz and Z. Komargodski, “The Exact Bremsstrahlung Function in  $N=2$  Superconformal Field Theories,” *Phys. Rev. Lett.* **116**, 081601 (2016) [arXiv:1510.01332 [hep-th]].
- [19] T. Nishioka and I. Yaakov, “Supersymmetric Renyi Entropy,” *JHEP* **1310**, 155 (2013) [arXiv:1306.2958 [hep-th]].
- [20] M. Crossley, E. Dyer and J. Sonner, “Super-Renyi entropy & Wilson loops for  $\mathcal{N}=4$  SYM and their gravity duals,” *JHEP* **1412**, 001 (2014) [arXiv:1409.0542 [hep-th]]; X. Huang and Y. Zhou, “ $\mathcal{N}=4$  Super-Yang-Mills on conic space as hologram of STU topological black hole,” *JHEP* **1502**, 068 (2015) [arXiv:1408.3393 [hep-th]].
- [21] N. Hama and K. Hosomichi, “Seiberg-Witten Theories on Ellipsoids,” *JHEP* **1209**, 033 (2012) [*JHEP* **1210**, 051 (2012)] [arXiv:1206.6359 [hep-th]].
- [22] T. Nataka and S. Terashima, “Supersymmetric Gauge Theories on a Squashed Four-Sphere,” *JHEP* **1312**, 001 (2013) [arXiv:1310.5939 [hep-th]].
- [23] L. F. Alday, P. Richmond and J. Sparks, “The holographic supersymmetric Renyi entropy in five dimensions,” *JHEP* **1502**, 102 (2015) [arXiv:1410.0899 [hep-th]]; N. Hama, T. Nishioka and T. Ugajin, “Supersymmetric Renyi entropy in five dimensions,” *JHEP* **1412**, 048 (2014) [arXiv:1410.2206 [hep-th]].
- [24] H. C. Kim, J. Kim and S. Kim, “Instantons on the 5-sphere and M5-branes,” arXiv:1211.0144 [hep-th]; J. Kallen and M. Zabzine, “Twisted supersymmetric 5D Yang-Mills theory and contact geometry,” *JHEP* **1205**, 125 (2012) [arXiv:1202.1956 [hep-th]]; K. Hosomichi, R. K. Seong and S. Terashima, “Supersymmetric Gauge Theories on the Five-Sphere,” *Nucl. Phys. B* **865**, 376 (2012) [arXiv:1203.0371 [hep-th]]; J. Kallen, J. Qiu and M. Zabzine, “The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere,” *JHEP* **1208**, 157 (2012) [arXiv:1206.6008 [hep-th]]; H. C. Kim and S. Kim, *JHEP* **1305**, 144 (2013) [arXiv:1206.6339 [hep-th]]. G. Lockhart and C. Vafa, “Superconformal Partition Functions and Non-perturbative Topological Strings,” arXiv:1210.5909 [hep-th]; Y. Imamura, “Perturbative partition function for squashed  $S^5$ ,” *PTEP* **2013**, no. 7, 073B01 (2013) [arXiv:1210.6308 [hep-th]].
- [25] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in  $N=2$  supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **426**, 19 (1994) Erratum: [*Nucl. Phys. B* **430**, 485 (1994)] [hep-th/9407087], “Monopoles, duality and chiral symmetry breaking in  $N=2$  supersymmetric QCD,” *Nucl.*

- Phys. B **431**, 484 (1994) [hep-th/9408099].
- [26] A. Grassi, M. Marino and S. Zakany, “Resumming the string perturbation series,” JHEP **1505**, 038 (2015) [arXiv:1405.4214 [hep-th]].
  - [27] M. Shifman, “Resurgence, operator product expansion, and remarks on renormalons in supersymmetric Yang-Mills theory,” J. Exp. Theor. Phys. **120**, no. 3, 386 (2015) [arXiv:1411.4004 [hep-th]].
  - [28] J. Gomis, T. Okuda and V. Pestun, “Exact Results for ’t Hooft Loops in Gauge Theories on  $S^4$ ,” JHEP **1205**, 141 (2012) [arXiv:1105.2568 [hep-th]].
  - [29] T. Azeanagi, M. Fujita and M. Hanada, “From the planar limit to M-theory,” Phys. Rev. Lett. **110**, no. 12, 121601 (2013) [arXiv:1210.3601 [hep-th]]; T. Azeanagi, M. Hanada, M. Honda, Y. Matsuo and S. Shiba, “A new look at instantons and large-N limit,” JHEP **1405**, 008 (2014) [arXiv:1307.0809 [hep-th]].
  - [30] By “Borel summability”, we mean Borel summability along positive real axis throughout this paper. However, if the localization formula is still correct for complex  $g$ , then our argument also shows Borel summability in right-half complex plane except squashed sphere partition functions.
  - [31] Strictly speaking, we prove that the perturbative series are convergent or asymptotic but Borel summable. Hence our result is consistent with expected convergence in planar limit [12]. We also assume that  $S^4$  partition function is well defined. Physically this corresponds to conformal field theory, its mass deformation and asymptotic free theory.
  - [32] We can also include FI-term and mass. Effects of the FI-term and mass are addition of a linear function of  $a$  in the exponent of  $Z_{cl}$  and a constant shift in one-loop and instanton parts, respectively. These do not spoil our proof as long as the mass is real.
  - [33] We are considering unit  $S^4$ . For  $S^4$  with radius  $r$ , the  $\Omega$ -background parameters are  $\epsilon_1 = \epsilon_2 = r^{-1}$ .
  - [34] This expression for  $SU(2)$  already appeared in [8].
  - [35] Below we simply refer to analytic continuation of formal Borel transformation as Borel transformation.
  - [36]  $G(1+z) = (2\pi)^{\frac{z}{2}} e^{-\frac{(1+z+\gamma z^2)}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}}$ .
  - [37] We are taking branch cut of  $\sqrt{z}$  in negative real axis.
  - [38] To get  $SU(N)$ , we shall strip decoupling  $U(1)$  part [16].
  - [39]  $b$  is real for ellipsoid [21] while  $b$  can be complex in the setup of [22].
  - [40] When we have Chern-Simons (CS) term, there is  $\text{tr} a^3$  term but this does not disturb our argument. Although it would be interesting to study perturbation by (inverse of) CS level as in 3d CS matter theory, this is beyond scope of this paper.